# VERSAL UNFOLDING OF A NILPOTENT LIÉNARD EQUILIBRIUM WITHIN THE ODD LIÉNARD FAMILY

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ABSTRACT. Although the technique of versal unfolding is developed and applied effectively to nilpotent equilibria, there are still great difficulties in studying the cases of higher codimension, referred to degenerate Bogdanov-Takens bifurcations, because those involved terms of higher degree produce more equilibria and hetero-(homo-)clinic loops. In this paper we discuss versal unfolding of a nilpotent Liénard equilibrium within the odd Liénard family. Such a restricted versal unfolding preserves the practical sense but involves less parameters. We prove that the nilpotent Liénard equilibrium is degenerate of codimension 2 in the odd Liénard family. Thus we use two parameters to display all possible bifurcations within the odd Liénard family such as pitchfork bifurcation, saddle-center bifurcation and homoclinic (heteroclinic) loop bifurcation.

## 1. INTRODUCTION

One of the most important mechanical systems is the well-known Liénard equation ([11, 15, 19])

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

where $g$ presents the restoring force and $f$ denotes the friction coefficient such that $f$, $g$ are continuous functions and $g(0) = 0$.

It is a significant task to discuss versal unfolding of a degenerate nilpotent Liénard equilibrium $O$ restricted within the Liénard family. Such a restricted versal unfolding preserves the practical sense but involves less parameters ([27, 28]). The classical Liénard mechanism (see e.g. [16, 19] or [29, Chapter 4, p.220]) requires in system (1.1) the function $f$ to be even and the function $g$ to be odd, which forces $b_2 = 0$ and confines system (1.1) to the form

$$\begin{aligned} \tfrac{dx}{dt} &= y, \\ \tfrac{dy}{dt} &= -(a_2x^2 + O(x^4))y - (b_3x^3 + O(x^5)), \end{aligned} \tag{1.2}$$

called the *even Liénard form* simply. Another type of Liénard systems, where both $f$ and $g$ are odd in (1.1), was considered in [21]. Such systems are of the form

$$\begin{aligned} \tfrac{dx}{dt} &= y && := P_0(x, y), \\ \tfrac{dy}{dt} &= -f_0(x)y - g_0(x) && := Q_0(x, y), \end{aligned} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 37G05; 37G10; 34C23; 34C20.
*Key words and phrases.* Liénard system; versal unfolding; normal form; degeneracy; bifurcation.

called the *odd Liénard form* correspondingly, where

$$f_0(x) = a_1 x + a_3 x^3 + O(x^5),$$
$$g_0(x) = b_1 x + b_3 x^3 + O(x^5)$$

with real constant $a_i$ s and $b_j$ s. Corresponding to the above opposite case, the degeneracy of the nilpotent equilibrium $O$ requires that $a_1 = b_1 = 0$ but $(a_3, b_3) \neq (0, 0)$ in $\mathbb{R}^2$ and therefore system (1.3) with a nilpotent equilibrium at $O$ is of the form

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -b_3 x^3 - a_3 x^3 y + O(|(x, y)|^5). \end{aligned} \tag{1.4}$$

System (1.4) also has a double vanished eigenvalues and, unlike the classical Bogdanov-Takens bifurcation case, the origin $O$ is not a cusp but either a saddle, center or focus by [4, Chapter 3, Theorem 3.5] or [29, Chapter 2, Theorem 7.2- 7.3]. More concretely, $O$ is a (nilpotent) saddle if $b_3 < 0$, or either a nilpotent center or a nilpotent focus if $b_3 > 0$ since $a_3 b_3 \neq 0$. Therefore, it is also interesting to discuss the restricted versal unfolding of the nilpotent equilibrium $O$ within the odd Liénard family (1.3) and the unfolding may exhibit bifurcations different from bifurcations of (1.2).

In this paper we investigate versal unfolding of the nilpotent Liénard equilibrium of system (1.4) within the odd Liénard family (1.3). This restricted versal unfolding cannot be deduced from any result of [5] because in [5] neither the degenerate system (when unfolding parameters equal zeros) nor the unfolding system near the nilpotent equilibrium is of the odd Liénard form. Moreover, the restricted versal unfolding also cannot be obtained with the well-known Bogdanov-Takens normal form because of the odevity in $f_0$ and $g_0$. In contrast to that the nilpotent equilibrium $O$ was specified to be a saddle or focus, we have to work in the case that the nilpotent equilibrium is a nilpotent saddle, a nilpotent focus or a nilpotent center. Besides, the lowest degree of system (1.4) is 3, which is higher than that of the even Lienard system (1.2) and makes difficulties in discussion. We will prove that the nilpotent Liénard equilibrium of system (1.4) is degenerate of codimension 2 in the odd Liénard family. Thus we can introduce two parameters to unfold the equilibrium versally within the odd Liénard family, displaying pitchfork bifurcation, saddle-center bifurcation and homoclinic (heteroclinic) loop bifurcation.

## 2. Versal Unfoldings

Let $\mathcal{L}_o$ consist of all planar $\mathcal{C}^4$ odd Liénard vector fields of form (1.3), which is a linear space and well defined in a compact neighborhood of the equilibrium $O$. Moreover, $\mathcal{L}_o$ can be regarded as a topological space with the topology induced from the maximum norm.

In order to give a versal unfolding of system (1.4) in $\mathcal{L}_o$, it suffices to consider its fourth order truncation of the form

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -b_3 x^3 - a_3 x^3 y \end{aligned} \tag{2.1}$$

near the origin, which is regarded as the principal system as in [1]. As known in [3], system (2.1) is degenerate of codimension greater than 2 at $O$. However, restricted within $\mathcal{L}_o$ the codimension of system (1.4) may be less. In $\mathcal{L}_o$, system (2.1) has a natural unfolding

$$\begin{array}{l} \frac{dx}{dt} = y := P(x,y), \\ \frac{dy}{dt} = \mu_1 x + \mu_2 xy + bx^3 + ax^3 y := Q(x,y) \end{array} \tag{2.2}$$

by keeping the structure of systems in $\mathcal{L}_o$, where $\mu = (\mu_1, \mu_2)$ denotes the tuple of the unfolding parameters near $(0,0)$ and we write $-b_3, -a_3$ as $b, a$ respectively for simpler notations. Notice that the unfolding system (2.2) has no constant terms since the origin is assumed always an equilibrium and $ab \neq 0$. Neither the term $x^2$ nor the term $y^2$ exists in the unfolding system (2.2) because of the odevity of functions $f$ and $g$ in system (1.1).

Before proving the versality of the unfolding system (2.2), we need to know the codimension of the degeneracy in the odd Liénard family $\mathcal{L}_o$. Let $V_0$ denote the degenerate system (2.1) and $\mathcal{L}_o(x)$ be the space of germs at the point $x = (x_1, x_2) \in \mathbb{R}^2$ of vector fields in the family $\mathcal{L}_o$. Fixed a neighborhood $U_0$ of the origin in $\mathbb{R}^2$, let

$$\mathcal{V} := \cup_{\xi \in U_0} \mathcal{L}_o(\xi),$$

which is a topological space defined as for the space of vector fields on a manifold. Each $\mathcal{L}_o(\xi)$ in $\mathcal{V}$ corresponds to a point $\xi \in \mathbb{R}^2$ and vector fields at the point. A germ $V_\xi \in \mathcal{V}$ at $\xi \in U_0$ defines a vector field of a planar odd Liénard system

$$\frac{dx}{dt} = V(x), \quad x \in U_\xi, \tag{2.3}$$

where $U_\xi \subset U_0$ is a neighborhood of $\xi$.

In order to give a versal unfolding for $V_0$ in $\mathcal{V}$, we need to describe the class of germs having the same singularity as $V_0$. This class is

$$S := \{V_\xi \in \mathcal{V} |\ V_\xi \text{ satisfies } (H_1),\ (H_1) \text{ and } (H_3)\},$$

where

- $(H_1)$**:** the linearization of $V_\xi(x)$ at $x = \xi$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;
- $(H_2)$**:** the coefficients of the terms of degree 2 in the expansion (1.4) of $V_\xi(x)$ always vanish;
- $(H_3)$**:** only two coefficients of the 3-order and 4-order terms $x^3$ and $x^3y$ in the expansion (1.4) of $V_\xi(x)$ are not equal to 0, i.e., $b_3 a_3 \neq 0$.

The conditions $(H_2)$ and $(H_3)$ guarantee that $V_\xi(x)$ belongs to the family $\mathcal{L}_o$ and has some degeneracy. Without $(H_3)$, additional degeneracy will be caused. The following lemma shows that $(H_3)$ is the condition of nondegeneracy for an unfolding of codimension 2.

**Lemma 1.** *The set $S$ is a smooth submanifold of codimension 2 near $V_0$ in $\mathcal{V}$.*

*Proof.* For a given $k \in \mathbb{Z}_+$, let $j^k V_\xi$ denote the $k$-jet of $V_\xi$ at $\xi$, i.e., the vector of all the coefficients in the $k$-th order Taylor expansion. Let $J^k = \{j^k V_\xi |\ V_\xi \in \mathcal{V}\}$. A natural projection $\pi_k : \mathcal{V} \to J^k$ can be defined by

$$V_\xi \mapsto (V(\xi), DV(\xi), \cdots, D^k V(\xi)),$$

where $V$ is defined in (2.3) and $D^k V(\xi)$ is the $k$-th order derivative of $V$ at $x = \xi$.

First of all, we prove that $\pi_1(S)$ constructs a smooth submanifold of codimension 2 near $\pi_1(V_0)$ in $J^1$. Note that the origin is an equilibrium for all considered systems. By the definition of $S$, we have

$$\begin{aligned} \pi_1(S) \quad = \quad & \left\{(0, DV(\xi))|\ \frac{\partial V_1(\xi)}{\partial x_1} \equiv 0, \frac{\partial V_1(\xi)}{\partial x_2} \equiv 1,\right. \\ & \left. \qquad\qquad \frac{\partial V_2(\xi)}{\partial x_1} = \frac{\partial V_2(\xi)}{\partial x_2} = 0\right\}, \end{aligned} \tag{2.4}$$

where $V_1$ and $V_2$ are components of $V$. The structure of the submanifold for $\pi_1(S)$ is observed from the projection $\pi_1$ to a finite-dimensional Euclidean space. The last two equalities in (2.4) confine the submanifold $\pi_1(S)$ to be of codimension 2 near $\pi_1(V_0)$ in $J^1$.

Next, we claim that for each $k \geq 2$ the set $\pi_k(S)$ is also a smooth submanifold of codimension 2 near $\pi_k(V_0)$ in $J^k$. The structure of the submanifold for $\pi_k(S)$ is observed similarly to the last step. Define a projection $\pi_{k1} : J^k \to J^1$ such that

$$(V(\xi), DV(\xi), \cdots, D^k V(\xi)) \mapsto (V(\xi), DV(\xi)),$$

which is clearly a regular submersion. Hence, the map $\pi_{k1}$ intersects $\pi_1(S) \subset J^1$ transversally. By Theorem 3.3 in [12, p. 22], $\pi_{k1}^{-1}(\pi_1(S))$ is a smooth submanifold in $J^k$ and the codimension of $\pi_{k1}^{-1}(\pi_1(S))$ in $J^k$ is the same as the codimension of $\pi_1(S)$ in $J^1$, i.e.,

$$\operatorname{codim} \pi_{k1}^{-1}(\pi_1(S)) = \operatorname{codim} \pi_1(S) = 2. \tag{2.5}$$

On the other hand, $\pi_k(S) \subset \pi_{k1}^{-1}(\pi_1(S))$. Actually, $\pi_k(S)$ consists of those in $\pi_{k1}^{-1}(\pi_1(S))$ with restriction $(H_3)$. Furthermore, $\pi_k(S)$ is an open subset of $\pi_{k1}^{-1}(\pi_1(S))$ near $\pi_k(V_0)$ because of the strict inequalities $(H_3)$. It follows from (2.5) that in $J^k$,

$$\operatorname{codim} \pi_k(S) = 2. \tag{2.6}$$

Since $\pi_k$ is a smooth submersion from $\mathcal{V}$ to $J^k$, we know that $\pi_k$ intersects $\pi_k(S) \subset J^k$ transversally. As above, Theorem 3.3 in [12] also implies that $S = \pi_k^{-1}(\pi_k(S))$ is a smooth manifold in $\mathcal{V}$ and

$$\operatorname{codim} S = \operatorname{codim} \pi_k^{-1}(\pi_k(S)) = \operatorname{codim} \pi_k(S) = 2$$

by (2.6). It means that $S$ is a smooth submanifold of codimension 2 in $\mathcal{V}$. $\square$

By Lemma 1, a universal unfolding of (2.1) in the family $\mathcal{L}_o$ is a system with two unfolding parameters and the parameterized system is a submanifold of dimension 2 intersecting $S$ transversally.

**Theorem 2.** *System (2.2) with condition* $(H_3)$ *is a versal unfolding of system (1.4) in* $\mathcal{L}_o$.

*Proof.* Let $V(\mu) := (P_1(x, y, \mu), Q_1(x, y, \mu))$, where $P_1$ and $Q_1$ denote the right-hand sides of the first equation and the second equation of (2.2) respectively. Clearly, $V(0) = V_0 \in S$. In order to prove the transversality of $V(\mu)$, define the map $\varphi : \mathbf{R}^2 \to J^3$ by

$$\mu \mapsto \pi_3(V(\mu)) = (V(\mu), DV(\mu), D^2V(\mu), D^3V(\mu)).$$

It suffices to prove that $\varphi$ intersects $\pi_3(S) \subset J^3$ transversally at $\pi_3(V_0)$. Consider an open neighborhood $\mathcal{U}$ of $\mu = 0$. By condition $(H_1)$, the Jacobian matrix $DV(\mu)$ is nilpotent at the intersection $\varphi(\mathcal{U}) \cap \pi_3(S)$, i.e.,

$$\left\{ \begin{array}{lcl} \frac{\partial}{\partial x} Q_1(x, y, \mu_1, \mu_2) & = & \mu_1 + \mu_2 y + 3bx^2 + 3ax^2 y = 0, \\ \frac{\partial}{\partial y} Q_1(x, y, \mu_1, \mu_2) & = & \mu_2 x + ax^3 = 0. \end{array} \right. \tag{2.7}$$

Furthermore, the Jacobian matrix of $\varphi$ at $\mu = 0$ contains a sub-matrix

$$\left[ \begin{array}{cc} \frac{\partial}{\partial \mu_1}(\frac{\partial Q_1}{\partial x}) & \frac{\partial}{\partial \mu_2}(\frac{\partial Q_1}{\partial x}) \\ \frac{\partial}{\partial \mu_1}(\frac{\partial Q_1}{\partial y}) & \frac{\partial}{\partial \mu_2}(\frac{\partial Q_1}{\partial y}) \end{array} \right]_{\mu=0} = \left[ \begin{array}{cc} 1 & y \\ 0 & x \end{array} \right],$$

which has rank 2 when $x \neq 0$. Therefore, the Jacobian matrix of $\varphi$ is of full rank when $x \neq 0$, implying the transversality of $\varphi$. Moreover, when $x = 0$ any unfolding of system (1.4) in the class $\mathcal{L}_o$ must have the form $\dot{x} = y, \dot{y} = 0$ because of the odevity in $f_0$ and $g_0$. And system (2.2) also has the form $\dot{x} = y, \dot{y} = 0$ if $x = 0$, implying the versality of system (2.2) at this case.

In order to give a versal unfolding of system (1.4), it suffices to consider its truncation of degree four. We would see that system (2.2) is a general unfolding of truncated (1.4) by preserving the structure of the family $\mathcal{L}_o$. Then system (2.2) is a versal unfolding of system (1.4). □

Remark that system (2.2), being an unfolding of system (1.4), is not only versal but also universal because it contains the least number of unfolding parameters.

## 3. Bifurcations

In this section we investigate the universal unfolding (2.2) for all local bifurcations in a neighborhood of the degenerate system (2.1) at equilibrium $O : (0, 0)$.

**Theorem 3.** *There exist at most three equilibria of system (2.2). The origin* $O : (0, 0)$ *is always an equilibrium of (2.2). When the unfolding parameter* $(\mu_1, \mu_2)$ *varies apart from* $(0, 0)$ *and through the curve*

$$\mathcal{C}_1 := \{(\mu_1, \mu_2)|\ \mu_1 = 0\},$$

*two equilibria* $A_\pm : (\pm\sqrt{-\mu_1/b}, 0)$ *of system (2.2) arise from a pitchfork bifurcation and* $O$ *remains an equilibrium if* $\mu_1 b < 0$. *Moreover, equilibrium* $O$ *of system (2.2) is a saddle if either* $\mu_1 > 0$ *or* $\mu_1 = 0$ *and* $b > 0$, *but a center in other cases.*

*Equilibria $A_\pm : (\pm\sqrt{-\mu_1/b}, 0)$ are either saddles if $\mu_1 < 0$ and $b > 0$, or sinks (stable foci or stable nodes) if $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b < 0$, or sources (unstable foci or unstable nodes) if $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b > 0$, or centers if $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b = 0$.*

*Proof.* It is not difficult to find that all equilibria lie on the $x$-axis from the first equation of system (2.2). Notice that the origin $O : (0, 0)$ always is an equilibrium.

Direct calculation shows that equilibria $A_\pm : (\pm\sqrt{-\mu_1/b}, 0)$ of system (2.2) appear from $O$ when $\mu_1 b < 0$. When $\mu_1 b \geq 0$, system (2.2) has a unique equilibrium, which is the origin $O : (0, 0)$. The number of equilibria from one becomes three when unfolding parameter $(\mu_1, \mu_2)$ passes through $\mathcal{C}_1$ and then a pitchfork bifurcation happens.

At $O : (0, 0)$, we can compute that the matrix of linear part of system (2.2) has eigenvalues $\pm\sqrt{\mu_1}$. Hence, if $\mu_1 > 0$, the eigenvalues are two reals with opposite signs, indicating that equilibrium $O$ is a saddle. In the case $\mu_1 < 0$, the eigenvalues become a pair of conjugate pure imaginary numbers, implying that $O$ is of center-focus type. Note that in system (2.2)

$$P(-x, y) = P(x, y), \ Q(-x, y) = -Q(x, y), \tag{3.1}$$

showing the symmetry of vector field (2.2) with respect to the $y$-axis if we do not consider the direction of vector field. Thus we get from [29, Chapter II.5] that $O$ is a center if $\mu_1 < 0$.

In the case $\mu_1 = 0$, equilibrium $O$ is a nilpotent degenerate singularity. We first consider the situation $\mu_2 \neq 0$. Thus we can use Theorem 3.5 of [4, Chapter III] or Theorem 7.2 of [29, Chapter II], which were given by desingularizing the degenerate equilibrium as shown in Section 7.2 of [9], to obtain that $O$ of system (2.2) is either a saddle if $\mu_2 \neq 0$ and $b > 0$, or a center if $\mu_2 \neq 0$ and $b < 0$ since (2.2) is symmetric with respect to the $y$-axis and $\mu_2^2 + 8b < 0$. When $\mu_1 = \mu_2 = 0$, applying Theorem 3.5 of [4] or Theorem 7.2 of [29] again, we get that equilibrium $O$ is a saddle if $b > 0$ and a center if $b < 0$ because of the symmetry of system (2.2).

When $\mu_1 b < 0$, two equilibria $A_\pm : (\pm\sqrt{-\mu_1/b}, 0)$ of system (2.2) appear. We only need to research the qualitative properties of equilibrium $A_+$ because system (2.2) is symmetric and then equilibrium $A_-$ has the same properties as $A_+$ after an opposite time rescaling. Moving equilibrium $A_+$ to the origin, we consider the trace $T_+$ and determinant $D_+$ of the matrix of system (2.2) at $A_+$. We calculate that

$$T_+ = \sqrt{-\frac{\mu_1}{b}}(\mu_2 - \frac{a\mu_1}{b}), \qquad D_+ = 2\mu_1. \tag{3.2}$$

Therefore, if $\mu_1 < 0$, the eigenvalues are two reals with opposite signs and $A_+$ is a saddle. When $\mu_1 > 0$ and $\mu_2 - \frac{a\mu_1}{b} > 0$ (resp. $< 0$), equilibrium $A_+$ is an unstable (resp. stable) focus or node.

When $b<0$, $\mu_1>0$ and $\mu_2-\frac{a\mu_1}{b}=0$, after a linear tranformation $\tilde{x}=\frac{x}{\sqrt{\mu_1}}$, $\tilde{y}=\frac{y}{\sqrt{2\mu_1}}$ and a time rescaling $dt=\frac{d\tilde{t}}{\sqrt{2\mu_1}}$, system (2.2) becomes

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x-\tfrac{3\sqrt{-b}}{2}x^2+\tfrac{b}{2}x^3+y\Big(-\tfrac{\sqrt{2}a\mu_1}{b}x+\tfrac{3a\mu_1}{\sqrt{-2b}}x^2+\tfrac{a\mu_1}{\sqrt{2}}x^3\Big)\end{aligned}$$

at $A_+$, where we still use $x$ amd $y$ to represent $\tilde{x}$ and $\tilde{y}$ for simplicity. Notice that the eigenvalues become a pair of conjugate pure imaginary numbers at $A_+$, implying that $A_+$ is of center-focus type. Applying Theorem 1 (b) of [8], equilibrium $A_+$ of system (2.2) is a center in this case. The proof is completed. □

**Theorem 4.** *When $\mu_1<0$ and $b>0$ there exists a heteroclinic loop which connects with the saddles $A_\pm$ and surrounds the center $O$. When $\mu_1>0$, $b<0$ and $\mu_2-a\mu_1/b=0$, there exist two homooclinic loops linking the saddle $O$ and surrounding centers $A_\pm$ respectively.*

*Proof.* In the case $\mu_1<0$ and $b>0$, equilibria $A_\pm$ are saddles and $O$ is a center by Theorem 3. Note that the Jacobian matrix of system (2.2) at $A_+$ has eigenvalues $(T_+\pm\sqrt{T_+^2-4D_+})/2$ corresponding to the eigenvectors $(2/(T_+\pm\sqrt{T_+^2-4D_+}),1)^T$, where $T_++\sqrt{T_+^2-4D_+}>0$, $T_+-\sqrt{T_+^2-4D_+}<0$ and $T_+,D_+$ are exhibited in (3.2). Moreover, noticing directions of vector field we obtain that $\dot{x}=y>0$ if $y>0$ and $\dot{x}<0$ if $y<0$. Therefore, the unstable (or stable) manifold of saddle $A_+$ will go to the $y$-axis and intersect the negative (or positive) $y$-axis at a point $B_1^+$ (or $B_2^+$) as the time $t$ increases (or decreases). Appying the symmetry of vector field (2.2) with respect to the $y$-axis, the stable (or unstable) manifold of saddle $A_-$ also intersects the negative (or positive) $y$-axis at the point $B_1^+$ (or $B_2^+$) as the time $t$ decreases (or increases). The uniqueness of solutions indicates the existence of a heteroclinic loop which connects with the saddles $A_\pm$ and surrounds the center $O$, as shown in Figure 3.

In the case $\mu_1>0$, $b<0$ and $\mu_2-a\mu_1/b=0$, equilibria $A_\pm$ become centers and $O$ becomes a saddle from Theorem 3. Thus, there exist two homooclinic loops linking the saddle $O$ and surrounding two centers $A_\pm$ respectively, which are the boundaries of the center fields of centers $A_\pm$, as shown in Figure 6. □

Remark that the heteroclinic loop linking two saddles $A_\pm$ disappears and three equilibria $A_\pm$ and $O$ coalesce at the origin, when $\mu_1$ varies from negative to zero. Then a bifurcation of heteroclinic loop happens with the pitchfork bifurcation. When $\mu_1>0$, $b<0$ and $\mu_2-a\mu_1/b\neq 0$, equilibria $A_\pm$ become stable or unstable, and the two homoclinic loops linking the saddle $O$ split and a new bigger homoclinic loop linking the saddle $O$ appears by the symmetry of vector field (2.2), as shown in Figures 4-5. Hence a bifurcation of homoclinic loop happens. When $\mu_1$ varies from positive to zero, the two homoclinic loops linking the saddle $O$ disappear and three equilibria $A_\pm$ and $O$ coalesce at the origin. Then a bifurcation of homoclinic loop also happens together with the pitchfork bifurcation.

## 4. Simulations and Remarks

We will illustrate our results by numerical simulations in the following figures. As $(\mu_1, \mu_2) = (0, 0)$, system (2.2) has a unique equilibrium $O$ by Theorem 3, which is a saddle if either $\mu_1 > 0$ or $\mu_1 = 0$ and $b > 0$ shown in Figure 1, and is a center in other cases shown in Figure 2, where the red points represent equilibria.

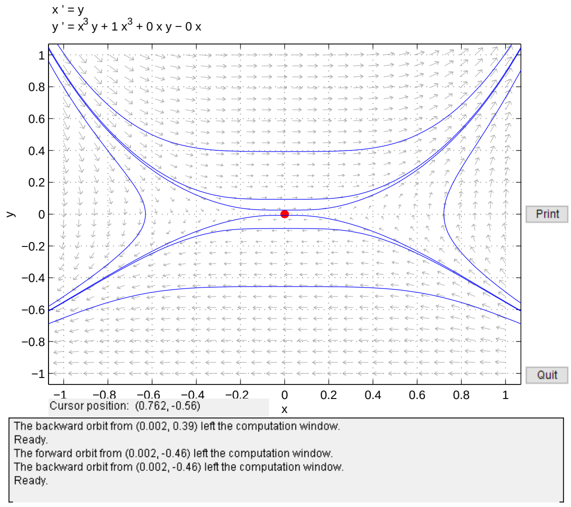


Figure 1. Equilibrium $O$ is a saddle as $\mu_1 > 0$ or $\mu_1 = 0$ and $b > 0$.

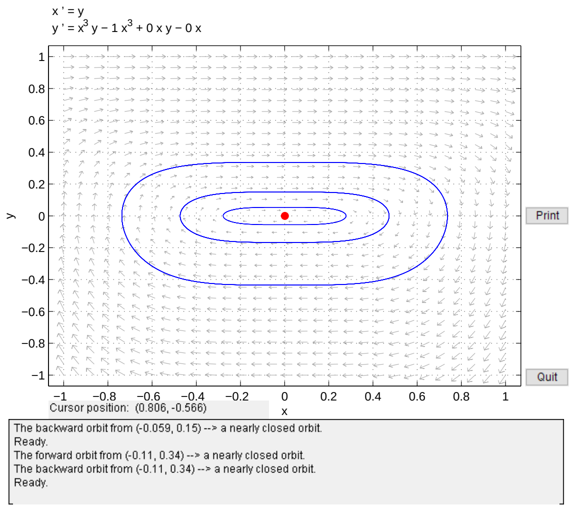


Figure 2. Equilibrium $O$ is a center as $\mu_1 < 0$ or $\mu_1 = 0$ and $b < 0$.

As $\mu = (\mu_1, \mu_2)$ varies from $\mathcal{C}_1$ (the bifurcation curve given in Theorem 3) into the region $\{\mu \in \mathbb{R} | \mu_1 > 0\}$ (or $\{\mu \in \mathbb{R} | \mu_1 < 0\}$), equilibrium $O$ changes from a degenerate saddle or center to a simple saddle or a simple center. At the same time, two equilibrium $A_\pm$ emerge, which can be saddles if $\mu_1 < 0$ and $b > 0$ shown in Figure 3, or foci (or nodes) if $\mu_1 > 0$, $b < 0$ and $\mu_2 + a\mu_1/b \neq 0$ shown in Figure 4 and Figure 5, or centers if $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b = 0$ shown in Figure 6.

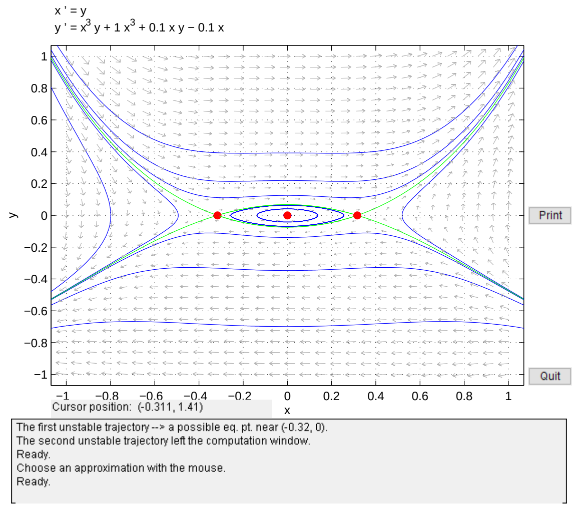


FIGURE 3. Equilibrium $O$ is a center while $A_\pm$ are saddle as $\mu_1 < 0$ and $b > 0$.

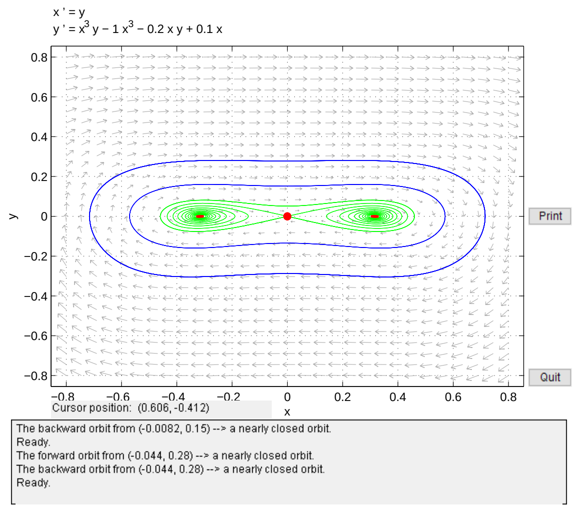


FIGURE 4. Equilibrium $O$ is a saddle while $A_+$ is a stable focus or node and $A_-$ is an unstable focus or node if $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b < 0$.

## ACKNOWLEDGEMENTS

The first author is partially supported by the National Natural Science Foundation of China (No. 11431008) and the European Union's Horizon 2020 research

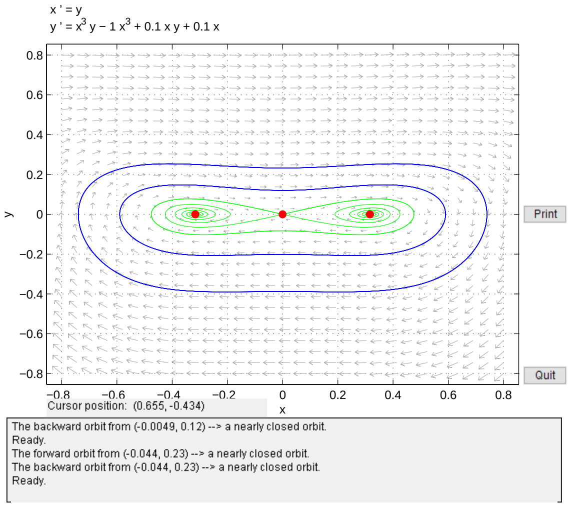


FIGURE 5. Equilibrium $O$ is a saddle while $A_+$ is an unstable focus or node and $A_-$ is a stable focus or node if $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b > 0$.

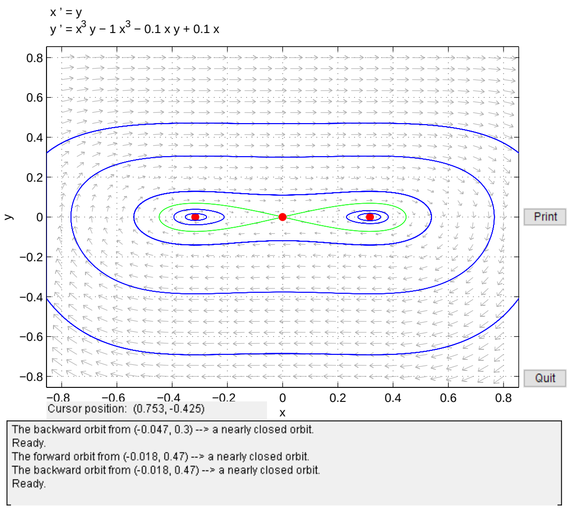


FIGURE 6. Equilibrium $O$ is a saddle while $A_\pm$ are centers as $\mu_1 > 0$, $b < 0$ and $\mu_2 - a\mu_1/b = 0$.

and innovation programme under the Marie Sklodowska-Curie grant agreement (No. 655212). The second author is supported by the National Natural Science Foundation of China (No. 11521061 & 11231001).

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